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Lagrangian formulation of electromagnetic fields in nondispersive medium by means of the extended Euler–Lagrange differential equation

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ABSTRACT

This work is concerned with the Lagrangian formulation of electromagnetic fields. Here, the extended Euler–Lagrange differential equation for continuous, nondispersive media is employed. The Lagrangian density for electromagnetic fields is extended to derive all four Maxwell's equations by means of electric and magnetic potentials. For the first time, ohmic losses for time and space variant fields are included. Therefore, a dissipation density function with time dependent and gradient dependent terms is developed. Both, the Lagrangian density and the dissipation density functions obey the extended Euler–Lagrange differential equation. Finally, two examples demonstrate the advantage of describing interacting physical systems by a single Lagrangian density.

1. Introduction

In 1788, Lagrange elegantly reformulated classical mechanics for conservative systems [1]. He reduced a large number of interacting forces in a system to only two forms of energy: the kinetic energy and the potential energy. Instead of dealing with vectors, only a scalar function must be considered. Furthermore, Lagrange's formalism holds in any coordinate system. These advantages initiated many works in mechanics and electronics using Lagrange's formalism.

For instance, a Lagrangian formulation for continuous systems in mechanics is given in [2]. José and Saletan [3] cover the Lagrangian formulation of continuum dynamics, and Scheck [4] treats discrete and continuous systems, the transition to a continuous system and the Hamilton variational principle for a continuous system.

In his works, Süsse [5–9] approaches theoretical foundations of electrical engineering using classical Lagrange and Hamilton formalisms, including losses. In [10,11] electrical lumped devices and electromechanical systems are described using Lagrange and Hamilton formalism with and without losses, whereas the generalized motion in Riemannian space, i.e. non-Euclidian, is considered. Further investigations on Lagrangians for lumped RLC-circuits are presented, for instance, in [12–14], and nonlinear, lumped RLC networks are described in [15,16].

Lagrangian formulation in electrodynamics has to consider time and local variations. For example, Kosyakov [17] constructs Lagrangian densities for a lossless electromagnetic field and for particles moving in such a field, whereas he focuses on developing Euler–Lagrange equations in tensor notation. Carroll [18] makes use of space–time reversed fields to construct a Lagrangian including ohmic loss. However, losses for stationary fields, i.e. for the case $\partial/\partial t = 0$, are not considered. Ter Haar [19] not only treats the Lagrangian density for continuous media without losses, but also derives two Maxwell's equations

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from the Lagrangian density for an electromagnetic field. Simonyi [20] derives the same two Maxwell's equations, namely Gauss' law $\nabla \cdot \mathbf{D} = \varrho_e$ and Ampère's law $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$. However, a complete treatment with respect to deriving all of Maxwell's equations is lacking.

In the following, a Lagrangian density will be established for electromagnetic fields in order to derive all four Maxwell's equations. Additionally, ohmic losses are considered which leads to the extended Euler–Lagrange differential equation and a dissipation density function. As a final point, two examples are presented which demonstrate that two interacting physical systems can be described by a single, scalar Lagrangian density.

2. Euler–Lagrange differential equation and Lagrangian density

Lagrange formalism in electromagnetics means, representing electromagnetic fields by energy densities and obtaining a Lagrangian density which obeys the so-called Euler–Lagrange differential equation. This differential equation fullfills the principle of least action. In electrodynamics, the fields are not only time dependent but also space dependent. Hence, the Euler–Lagrange differential equation for a continuous, lossless medium must be written as

$$\sum_{k=1}^{m} \frac{\partial}{\partial x_k} \left(\frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \eta_i}{\partial x_k} \right)} \right) + \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \eta_i}{\partial t} \right)} \right) - \frac{\partial \mathscr{L}}{\partial \eta_i} = \mathscr{F}_i, \tag{1}$$

where there is one equation for each value of i, and where $\mathscr{L}$ is the Lagrangian density, $\mathscr{F}_i$ is the external force corresponding to η_i, which is a time and space dependent potential, x_k are the system coordinates, and t is the time. The Lagrangian density, a scalar function, is of the form

$$\mathscr{L} = \mathscr{L}\left(\eta_i, \nabla \eta_i, \frac{\partial \eta_i}{\partial t} \right) \qquad i = 1, 2, \ldots, n. \tag{2}$$

Since the generalized forces $\mathscr{F}_i$ act only on η_i, the forces can be successively decoupled and are therefore included in the Lagrangian density as negative potential densities $-\eta_i \mathscr{F}_i$:

$$\sum_{k=1}^{m} \frac{\partial}{\partial x_k} \left(\frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \eta_i}{\partial x_k} \right)} \right) + \frac{\partial}{\partial t} \left(\frac{\partial \mathscr{L}}{\partial \left(\frac{\partial \eta_i}{\partial t} \right)} \right) - \frac{\partial \mathscr{L}}{\partial \eta_i} = 0, \tag{3}$$

with

$$\mathscr{L} = \mathscr{L}\left(\eta_i, \eta_i \mathscr{F}_i, \nabla \eta_i, \frac{\partial \eta_i}{\partial t} \right) \qquad i = 1, 2, \ldots, n. \tag{4}$$

The task is to setup the Lagrangian density (4) for the electromagnetic field obeying the Euler–Lagrange differential Eq. (3).

2.1. Lagrangian density for the electromagnetic field

The Lagrangian density consists of two parts, the kinetic part and the potential part. For an electromagnetic field, the kinetic part is the energy density stored in the electric field and the potential part is the energy density stored in the magnetic field. Energy densities in a linear electric field and a linear magnetic field are

$$w_e = \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \quad \text{and} \quad w_m = \frac{1}{2} \mathbf{B} \cdot \mathbf{H}, \tag{5}$$

respectively, where $\mathbf{D} = \varepsilon \mathbf{E}$ is the electric displacement, $\mathbf{E}$ is the electric field, $\mathbf{B} = \mu \mathbf{H}$ is the magnetic flux density, and $\mathbf{H}$ is the magnetic field. Electrical properties of the medium are described by the permittivity ε and permeability μ. For an electromagnetic field, the Lagrangian density in its contravariant form was discovered by Larmor [21] and is written as

$$\mathscr{L} = w_e - w_m = \frac{1}{2} \varepsilon \mathbf{E}^2 - \frac{1}{2} \mu \mathbf{H}^2. \tag{6}$$

The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ can be expressed by potential functions φ and $\mathbf{A}$ [22]. Therefore, the Lagrangian density is conveniently written as

$$\mathscr{L} = \frac{1}{2} \varepsilon \left(\nabla \varphi + \frac{\partial \mathbf{A}}{\partial t} \right)^2 - \frac{1}{2\mu} (\nabla \times \mathbf{A})^2, \tag{7}$$

with

$$\mathbf{E} = -\nabla \varphi - \frac{\partial \mathbf{A}}{\partial t} \tag{8}$$

$$\mu \mathbf{H} = \mathbf{B} = \nabla \times \mathbf{A}, \tag{9}$$

where φ and $\mathbf{A}$ are the electric scalar potential and the magnetic vector potential, respectively.

We can identify the potentials η_i in Eq. (3) with $\eta_1 = \varphi$, $\eta_2 = A_x$, $\eta_3 = A_y$ and $\eta_4 = A_z$, where x, y, and z are the Cartesian coordinates. The sources for an electromagnetic field are the electric volume charge density $-\varrho_e = \mathscr{F}_1$ and the electric current density vector $\mathbf{J_{e0}}$ with $J_{e0,x} = \mathscr{F}_2, J_{e0,y} = \mathscr{F}_3$, and $J_{e0,z} = \mathscr{F}_4$. With these sources, the Lagrangian density in Eq. (7) can be extended to

$$\mathscr{L} = \frac{1}{2}\varepsilon\left(\nabla\varphi + \frac{\partial\mathbf{A}}{\partial t}\right)^2 - \frac{1}{2\mu}(\nabla\times\mathbf{A})^2 - \varphi\varrho_e + \mathbf{J_{e0}}\cdot\mathbf{A}. \tag{10}$$

Now, it must be shown that the Lagrangian density given by Eq. (10) results in Maxwell's equations. Substituting $\eta_1 = \varphi$ in Eq. (3) and plugging $\mathscr{L}$ into the Euler–Lagrange differential Eq. (3) results in

$$\begin{aligned}
&\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)} = \varepsilon\left(\frac{\partial\varphi}{\partial x} + \frac{\partial A_x}{\partial t}\right) = -D_x\\
&\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial y}\right)} = \varepsilon\left(\frac{\partial\varphi}{\partial y} + \frac{\partial A_y}{\partial t}\right) = -D_y\\
&\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial z}\right)} = \varepsilon\left(\frac{\partial\varphi}{\partial z} + \frac{\partial A_z}{\partial t}\right) = -D_z\\
&\Rightarrow \sum_{k=1}^{3}\frac{\partial}{\partial x_k}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)} = -\nabla\cdot\mathbf{D}\\
&\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} = 0\\
&\frac{\partial\mathscr{L}}{\partial\varphi} = -\varrho_e.
\end{aligned} \tag{11}$$

By inspection, this is Maxwell's first equation, also called Gauss' law:

$$\nabla\cdot\mathbf{D} = \varrho_e. \tag{12}$$

Further, substituting $\eta_2 = A_x$ in Eq. (3) and plugging $\mathscr{L}$ into the Euler–Lagrange differential equation results in

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_x}{\partial x}\right)} = 0\\
&\frac{\partial}{\partial y}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_x}{\partial y}\right)} = \frac{\partial}{\partial y}\frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = +\frac{\partial}{\partial y}H_z\\
&\frac{\partial}{\partial z}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_x}{\partial z}\right)} = \frac{\partial}{\partial z}\frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = -\frac{\partial}{\partial z}H_y\\
&\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_x}{\partial t}\right)} = \frac{\partial}{\partial t}\varepsilon\left(\frac{\partial\varphi}{\partial x} + \frac{\partial A_x}{\partial t}\right) = -\frac{\partial}{\partial t}D_x\\
&\frac{\partial\mathscr{L}}{\partial A_x} = J_{e0,x}
\end{aligned} \tag{13}$$

same for $\eta_3 = A_y$:

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_y}{\partial x}\right)} = \frac{\partial}{\partial x}\frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -\frac{\partial}{\partial x}H_z\\
&\frac{\partial}{\partial y}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_y}{\partial y}\right)} = 0\\
&\frac{\partial}{\partial z}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_y}{\partial z}\right)} = \frac{\partial}{\partial z}\frac{1}{\mu}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = +\frac{\partial}{\partial z}H_x\\
&\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial A_y}{\partial t}\right)} = \frac{\partial}{\partial t}\varepsilon\left(\frac{\partial\varphi}{\partial y} + \frac{\partial A_y}{\partial t}\right) = -\frac{\partial}{\partial t}D_y\\
&\frac{\partial\mathscr{L}}{\partial A_y} = J_{e0,y}
\end{aligned} \tag{14}$$

And likewise for $\eta_4 = A_z$:

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial(\frac{\partial A_z}{\partial x})} = \frac{\partial}{\partial x}\frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = +\frac{\partial}{\partial x}H_y \\
&\frac{\partial}{\partial y}\frac{\partial \mathscr{L}}{\partial(\frac{\partial A_z}{\partial y})} = \frac{\partial}{\partial y}\frac{1}{\mu}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) = -\frac{\partial}{\partial y}H_x \\
&\frac{\partial}{\partial z}\frac{\partial \mathscr{L}}{\partial(\frac{\partial A_z}{\partial z})} = 0 \\
&\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial(\frac{\partial A_z}{\partial t})} = \frac{\partial}{\partial t}\varepsilon\left(\frac{\partial \varphi}{\partial z} + \frac{\partial A_z}{\partial t}\right) = -\frac{\partial}{\partial t}D_z \\
&\frac{\partial \mathscr{L}}{\partial A_z} = J_{e0,z}
\end{aligned} \tag{15}$$

Bringing all sets of Eqs. (13)–(15) into vector form and rearranging the terms, yields Maxwell's second equation, also called Ampère's law:

$$\nabla \times \mathbf{H} = \mathbf{J_{e0}} + \frac{\partial \mathbf{D}}{\partial t}. \tag{16}$$

As shown, only two of the four Maxwell's equations can be derived by means of the Lagrangian density defined in Eq. (10). To obtain the other two Maxwell's equations the dual Lagrangian density is introduced.

2.2. Dual Lagrangian density for the electromagnetic field

Dirac [23] argued that there are magnetic monopoles, analogous to electric charges. Accepting the existence of magnetic monopoles, would yield a symmetric set of Maxwell's equations as presented, for instance, in [22]. As in [24,25], using the duality theorem, the magnetic field and the electric displacement are now defined as

$$\mathbf{H} = -\nabla\varphi_m - \frac{\partial \mathbf{A}_e}{\partial t}, \tag{17}$$

and

$$\mathbf{D} = -\nabla \times \mathbf{A}_e, \tag{18}$$

where φ_m is the magnetic scalar potential and $\mathbf{A}_e$ is the electric vector potential. This leads to the dual Lagrangian density

$$\begin{aligned}
\mathscr{L}_d &= \frac{1}{2}\mu\mathbf{H}^2 - \frac{1}{2}\varepsilon\mathbf{E}^2 \\
&= \frac{1}{2}\mu\left(\nabla\varphi_m + \frac{\partial \mathbf{A}_e}{\partial t}\right)^2 - \frac{1}{2\varepsilon}(\nabla \times \mathbf{A}_e)^2
\end{aligned} \tag{19}$$

Assuming a magnetic volume charge density ϱ_m and a magnetic current density $\mathbf{J_{m0}}$, the dual Lagrangian density can be extended to

$$\mathscr{L}_d = \frac{1}{2}\mu\left(\nabla\varphi_m + \frac{\partial \mathbf{A}_e}{\partial t}\right)^2 - \frac{1}{2\varepsilon}(\nabla \times \mathbf{A}_e)^2 - \varphi_m\varrho_m - \mathbf{J_{m0}} \cdot \mathbf{A_e}. \tag{20}$$

In line with Section 2.1, it can be shown that the dual Lagrange density leads to the other two equations of Maxwell. Substituting $\eta_5 = \varphi_m$ in Eq. (3) and plugging $\mathscr{L}_d$ into the Euler–Lagrange differential Eq. (3), results in Maxwell's third equation, also called Gauss' law for magnetization:

$$\nabla \cdot \mathbf{B} = \varrho_m. \tag{21}$$

Since magnetic monopoles have not been detected yet, the external force ϱ_m can be set to zero, and the conventional form $\nabla \cdot \mathbf{B} = 0$ is obtained.

Furthermore, substituting $\eta_6 = A_{e,x}$, $\eta_7 = A_{e,y}$, and $\eta_8 = A_{e,z}$ in Eq. (3) and plugging $\mathscr{L}_d$ into the Euler–Lagrange differential equation, leads to Maxwell's fourth equation in vector form, also called Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\mathbf{J_{m0}} - \frac{\partial \mathbf{B}}{\partial t}. \tag{22}$$

Again, when there is no magnetic current $\mathbf{J_{m0}}$, the conventional form $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ is obtained.

2.3. Total Lagrangian density

The sum of the Lagrangian density and the dual Lagrangian density is the total Lagrangian density $\mathscr{L}_{\mathrm{T}}$. All four of Maxwell's equations can be derived from this single, total Lagrangian density (23).

$$\begin{aligned}\mathscr{L}_{\mathrm{T}} = & \frac{1}{2}\left[\varepsilon\left(\nabla\varphi+\frac{\partial\mathbf{A}}{\partial t}\right)^2+\mu\left(\nabla\varphi_{\mathrm{m}}+\frac{\partial\mathbf{A_e}}{\partial t}\right)^2\right] \\ & -\frac{1}{2}\left[\frac{(\nabla\times\mathbf{A})^2}{\mu}+\frac{(\nabla\times\mathbf{A_e})^2}{\varepsilon}\right] \\ & -\varphi\varrho_{\mathrm{e}}+\mathbf{J_{e0}}\cdot\mathbf{A}-\varphi_{\mathrm{m}}\varrho_{\mathrm{m}}-\mathbf{J_{m0}}\cdot\mathbf{A_e}\end{aligned} \tag{23}$$

3. Extended Euler–Lagrange differential equation: losses

Whenever a medium is present, losses are introduced. Therefore, a dissipation density function $\mathscr{D}$ has to be considered, similar to the Lagrangian density. The Lagrangian density is an energy density whereas the dissipation density function is a power density. The dissipation density function is the time average rate of energy dissipation per unit volume $\partial w_{\mathrm{h}}/\partial t$, with w_{h} as the energy density dissipated in heat, and the energy dissipation due to EM-field variation in space. In the same way as a potential ψ must satisfy the well-known three-dimensional wave equation

$$\sum_{k=1}^{3} c^2\frac{\partial^2\psi}{\partial x_k^2}-\frac{\partial\psi^2}{\partial t^2}=0, \tag{24}$$

with $c=1/\sqrt{\varepsilon\mu}$ as the speed of light, the dissipation density function $\mathscr{D}$ has to satisfy

$$\sum_{k=1}^{\mathrm{m}}\frac{\partial\mathscr{D}}{\partial\left(c\frac{\partial\eta_i}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\eta_i}{\partial t}\right)}=0, \tag{25}$$

when there are no losses. However, considering losses, the right hand side of Eq. (25) is not equal to zero. Therefore, the Euler–Lagrange differential equation has to be extended with a generalized velocity dependent term $\partial\mathscr{D}/\partial\left(\frac{\partial\eta_i}{\partial t}\right)$ and a gradient dependent term $\frac{1}{c}\sum_{k=1}^{\mathrm{m}}\frac{\partial\mathscr{D}}{\partial(\partial\eta_i/\partial x_k)}$. Then, the extended Euler–Lagrange differential equation is written as

$$\sum_{k=1}^{\mathrm{m}}\frac{\partial}{\partial x_k}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta_i}{\partial x_k}\right)}\right)+\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta_i}{\partial t}\right)}\right)+\frac{1}{c}\sum_{k=1}^{\mathrm{m}}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\eta_i}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\eta_i}{\partial t}\right)}-\frac{\partial\mathscr{L}}{\partial\eta_i}=0. \tag{26}$$

It has to be mentioned that the external forces $\mathscr{F}_i$ can be included either as negative potential densities in the Lagrangian density or as negative loss densities in the dissipation density function. The first case has been preferred in this work, as shown in Sections 2.1 and 2.2.

3.1. Losses due to electric conductivity – the dissipation density function

In case of electric conductivity the losses are known as ohmic losses. In electromagnetics, Ohm's law is defined as

$$\mathbf{J}=\sigma_{\mathrm{e}}\mathbf{E}, \tag{27}$$

where $\mathbf{J}$ is the current density vector, σ_{e} is the electric conductivity of the medium and $\mathbf{E}$ is the electric field causing charge transport. Losses due to electric conductivity depend, as the electric field, on $\nabla\varphi$ and on $\partial\mathbf{A}/\partial t$. Hence, the dissipation density function, plugged into the Extended Euler–Lagrange differential Eq. (26), should yield the additive term given by (27) in Maxwell's second equation. The ansatz to find $\mathscr{D}$ is

$$\frac{1}{c}\sum_{k=1}^{3}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_x}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_x}{\partial t}\right)}=\sigma_{\mathrm{e}}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial A_x}{\partial t}\right) \tag{28}$$

$$\frac{1}{c}\sum_{k=1}^{3}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_y}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_y}{\partial t}\right)}=\sigma_{\mathrm{e}}\left(\frac{\partial\varphi}{\partial y}+\frac{\partial A_y}{\partial t}\right) \tag{29}$$

$$\frac{1}{c}\sum_{k=1}^{3}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_z}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial A_z}{\partial t}\right)}=\sigma_{\mathrm{e}}\left(\frac{\partial\varphi}{\partial z}+\frac{\partial A_z}{\partial t}\right) \tag{30}$$

$$\frac{1}{c}\sum_{k=1}^{3}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}=0. \tag{31}$$

However, these partial differential equations are coupled with respect to the spatial derivatives $\partial\varphi/\partial x_k$. To decouple these equations, the well-known Lorenz gauge

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0, \tag{32}$$

is used. Hence, Eq. (31) can be rewritten as

$$\frac{1}{c}\sum_{k=1}^{3}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial x_k}\right)}+\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}=\sigma_{\mathrm{e}}\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right). \tag{33}$$

Integrating both sides with respect to spatial and time derivatives

$$\frac{1}{c}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)}=\sigma_{\mathrm{e}}\frac{\partial A_x}{\partial x},\quad \frac{1}{c}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial y}\right)}=\sigma_{\mathrm{e}}\frac{\partial A_y}{\partial y},$$
$$\frac{1}{c}\frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial z}\right)}=\sigma_{\mathrm{e}}\frac{\partial A_z}{\partial z},\quad \frac{\partial\mathscr{D}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)}=\frac{\sigma_{\mathrm{e}}}{c^2}\frac{\partial\varphi}{\partial t},$$

correspondingly, leads to

$$\mathscr{D}_\varphi=c\sigma_{\mathrm{e}}\left[\frac{\partial\varphi}{\partial x}\frac{\partial A_x}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial A_y}{\partial y}+\frac{\partial\varphi}{\partial z}\frac{\partial A_z}{\partial z}\right]+\frac{\sigma_{\mathrm{e}}}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2. \tag{34}$$

The dissipation density function given by Eq. (34) satisfies Eqs. (28)–(30) except for the terms $\sigma_{\mathrm{e}}\partial A_x/\partial t$, $\sigma_{\mathrm{e}}\partial A_y/\partial t$ and $\sigma_{\mathrm{e}}\partial A_z/\partial t$. Plugging $\mathscr{D}_\varphi$ into these equations yields only $\sigma_{\mathrm{e}}\nabla\varphi$. Integrating $\frac{\partial\mathscr{D}}{\partial(\partial A_i/\partial t)}=\sigma_{\mathrm{e}}\frac{\partial A_i}{\partial t}$ on both sides, with $i=x,y,z$, leads to the time dependent term $\mathscr{D}_A=\frac{1}{2}\sigma_{\mathrm{e}}\left[(\partial A_x/\partial t)^2+(\partial A_y/\partial t)^2+(\partial A_z/\partial t)^2\right]$. The complete dissipation density function is then the sum of $\mathscr{D}_\varphi$ and $\mathscr{D}_A$:

$$\begin{aligned}\mathscr{D}=\;&c\sigma_{\mathrm{e}}\left[\frac{\partial\varphi}{\partial x}\frac{\partial A_x}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial A_y}{\partial y}+\frac{\partial\varphi}{\partial z}\frac{\partial A_z}{\partial z}\right]\\&+\frac{1}{2}\frac{\sigma_{\mathrm{e}}}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2\\&+\frac{1}{2}\sigma_{\mathrm{e}}\left[\left(\frac{\partial A_x}{\partial t}\right)^2+\left(\frac{\partial A_y}{\partial t}\right)^2+\left(\frac{\partial A_z}{\partial t}\right)^2\right],\end{aligned} \tag{35}$$

which satisfies Eqs. (28)–(31). The first three terms of Eq. (35) are gradient dependent terms and describe losses due to local field variations. The last four terms of Eq. (35) are time dependent terms and are analogous to the generalized velocity dependent (Rayleigh) dissipation function

$$\frac{\partial D}{\partial\dot{q}}=R\dot{q}\quad\Rightarrow\quad D=\frac{1}{2}R\dot{q}^2$$

for lumped elements, as in [10–12], where R and $\dot{q}$ are resistance and current, respectively.

3.2. Losses due to magnetic conductivity – the dual dissipation density function

Just as there is a dual Lagrangian density, there is a dual dissipation density function $\mathscr{D}_{\mathrm{d}}$. Similar to electric conductivity, a magnetic conductivity σ_{m} can be defined [26]. Due to the magnetic field the magnetic conductivity causes a magnetic current density

$$\mathbf{J}_{\mathrm{m}}=\sigma_{\mathrm{m}}\mathbf{H}. \tag{36}$$

The dual dissipation density function can be established in the same manner as the dissipation density function in Section 3.1. It is given as

$$\mathscr{D}_{\mathrm{d}}=c\sigma_{\mathrm{m}}\left[\frac{\partial\varphi_{\mathrm{m}}}{\partial x}\frac{\partial A_{\mathrm{e},x}}{\partial x}+\frac{\partial\varphi_{\mathrm{m}}}{\partial y}\frac{\partial A_{\mathrm{e},y}}{\partial y}+\frac{\partial\varphi_{\mathrm{m}}}{\partial z}\frac{\partial A_{\mathrm{e},z}}{\partial z}\right]+\frac{1}{2}\frac{\sigma_{\mathrm{m}}}{c^2}\left(\frac{\partial\varphi_{\mathrm{m}}}{\partial t}\right)^2+\frac{1}{2}\sigma_{\mathrm{m}}\left[\left(\frac{\partial A_{\mathrm{e},x}}{\partial t}\right)^2+\left(\frac{\partial A_{\mathrm{e},y}}{\partial t}\right)^2+\left(\frac{\partial A_{\mathrm{e},z}}{\partial t}\right)^2\right], \tag{37}$$

with the Lorenz gauge

$$\nabla\cdot\mathbf{A}_{\mathrm{e}}+\frac{1}{c^2}\frac{\partial\varphi_{\mathrm{m}}}{\partial t}=0. \tag{38}$$

3.3. Total dissipation density function

Analogous to the total Lagrangian density, the total dissipation density function is the sum of Eqs. (35) and (37):

$$\mathscr{D}_{\mathrm{T}} = \mathscr{D} + \mathscr{D}_{\mathrm{d}}. \tag{39}$$

Plugging $\mathscr{L}_{\mathrm{T}}$ and $\mathscr{D}_{\mathrm{T}}$ into the extended Euler–Lagrange differential equation yields all four of Maxwell's equations, including electric and magnetic losses in Maxwell's second and fourth equations:

$$\nabla \times \mathbf{E} = -\mathbf{J_{m0}} - \sigma_{\mathrm{m}}\mathbf{H} - \frac{\partial \mathbf{B}}{\partial t} \tag{40}$$

$$\nabla \times \mathbf{H} = \mathbf{J_{e0}} + \sigma_{\mathrm{e}}\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}. \tag{41}$$

4. Examples

4.1. Lagrangian density for a non-relativistic, charged particle interacting with an EM-field

The Lagrangian density for interacting systems is the sum of the Lagrangian density for the first system, the Lagrangian density for the second system and a Lagrangian density $\mathscr{L}_i$ accounting for the interacting part [3,27]. In this example, a charged particle of mass m and electric charge q moves in an electromagnetic field. By means of the Lagrangian density and Euler–Lagrange differential equation, the equations of motions for the particle are obtained. The particle is described by the Lagrangian density $\mathscr{L}_{\mathrm{mass}}$, and the electromagnetic field is described by the Lagrangian density $\mathscr{L}_{\mathrm{EM}}$.

$$\mathscr{L} = \mathscr{L}_{\mathrm{mass}} + \mathscr{L}_{\mathrm{EM}} + \mathscr{L}_i. \tag{42}$$

In [5], for instance, this system has been described by a Lagrangian. To obtain the Lagrangian density, the Lagrangian in [5] has to be derived with respect to volume V:

$$\mathscr{L} = \frac{\partial L}{\partial V} = \frac{\partial^3}{\partial z \partial y \partial x}\left(\frac{1}{2}m\dot{\mathbf{r}}^2 - q\varphi + q\,\dot{\mathbf{r}}\,\mathbf{A}\right), \tag{43}$$

with m, q, and $\dot{\mathbf{r}} = \dot{\mathbf{r}}(x, y, z)$ as the mass, charge and velocity vector of the particle, respectively. The potentials φ and $\mathbf{A}$ describe the electromagnetic field. This results in the three terms of the Lagrangian density. The mass related part is

$$\mathscr{L}_{\mathrm{mass}} = \frac{1}{2}\frac{\partial^3 m}{\partial z \partial y \partial x}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}\rho_{\mathrm{M}}\dot{\mathbf{r}}^2, \tag{44}$$

where ρ_{M} is the specific mass of the particle. The part describing the electromagnetic field is

$$\mathscr{L}_{\mathrm{EM}} = -\frac{\partial^3 q}{\partial z \partial y \partial x}\varphi - \frac{\partial^2 q}{\partial y \partial x}\frac{\partial \varphi}{\partial z} - \frac{\partial^2 q}{\partial z \partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial^2 q}{\partial z \partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial q}{\partial x}\frac{\partial^2 \varphi}{\partial z \partial y} - \frac{\partial q}{\partial y}\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial q}{\partial z}\frac{\partial^2 \varphi}{\partial y \partial x} - q\frac{\partial^3 \varphi}{\partial z \partial y \partial x}, \tag{45}$$

where $\partial^3 q/\partial z \partial y \partial x$ is the electric volume charge density ϱ_{e}. The part of the Lagrangian density which accounts for the interaction is derived as

$$\mathscr{L}_i = \varrho_{\mathrm{e}}\left(\dot{\mathbf{r}}\cdot\mathbf{A}\right) + \frac{\partial^2 q}{\partial y \partial x}\left(\dot{\mathbf{r}}\frac{\partial \mathbf{A}}{\partial z}\right) + \frac{\partial^2 q}{\partial z \partial x}\left(\dot{\mathbf{r}}\frac{\partial \mathbf{A}}{\partial y}\right) + \frac{\partial^2 q}{\partial z \partial y}\left(\dot{\mathbf{r}}\frac{\partial \mathbf{A}}{\partial x}\right) + \frac{\partial q}{\partial x}\left(\dot{\mathbf{r}}\frac{\partial^2 \mathbf{A}}{\partial z \partial y}\right) + \frac{\partial q}{\partial y}\left(\dot{\mathbf{r}}\frac{\partial^2 \mathbf{A}}{\partial z \partial x}\right) + \frac{\partial q}{\partial z}\left(\dot{\mathbf{r}}\frac{\partial^2 \mathbf{A}}{\partial y \partial x}\right) + q\left(\dot{\mathbf{r}}\frac{\partial^3 \mathbf{A}}{\partial z \partial y \partial x}\right). \tag{46}$$

Additionally assuming losses of the moving particle leads to the velocity dependent (Rayleigh) dissipation density function

$$\frac{\partial \mathscr{D}}{\partial \dot{r}_i} = \frac{\partial^3 \mathfrak{B}}{\partial z \partial y \partial x}\dot{r}_i \qquad \Rightarrow \qquad \mathscr{D} = \frac{1}{2}\mathfrak{b}\dot{\mathbf{r}}^2, \tag{47}$$

with $\mathfrak{B}$ and $\mathfrak{b}$ as friction and friction volume density, respectively, and $\dot{r}_i$ as the vector components of the velocity vector of the particle. Substituting the generalized coordinates $\eta_i = x,\ y,\ z,\ q,\ \varphi,\ A_x,\ A_y,\ A_z$ in the extended Euler–Lagrange Eq. (26) yields the equations of motion for the particle in density form:

$$\dot{\mathbf{r}}\frac{\partial^3 \mathbf{A}}{\partial z \partial y \partial x} - \frac{\partial^3 \varphi}{\partial z \partial y \partial x} = 0, \quad \varrho_{\mathrm{e}} = 0, \quad \varrho_{\mathrm{e}}\dot{\mathbf{r}} = 0, \tag{48}$$

and

$$\rho_{\mathrm{M}}\ddot{r}_i = -\frac{\partial^3}{\partial z \partial y \partial x}\left(q\frac{\partial A_i}{\partial t}\right) + \frac{\partial^3}{\partial z \partial y \partial x}\left(q\sum_{\mathrm{k}=1}^{3}\dot{r}_{\mathrm{k}}\frac{\partial A_{\mathrm{k}}}{\partial r_i}\right) - \frac{\partial^3}{\partial z \partial y \partial x}\left(q\frac{\partial \varphi}{\partial r_i}\right) - \mathfrak{b}\,\dot{r}_i, \tag{49}$$

with $i = 1, 2, 3$. Except for the losses $\mathfrak{b}\,\dot{r}_i$, Eqs. (48) and (49) are analogous to the equations of motion as given, for instance, in [5]. Integrating Eqs. (48) and (49) with respect to the volume $\partial^3/\partial z\partial y\partial x$ yields

$$\dot{\mathbf{r}}\mathbf{A} - \varphi = 0, \quad q = 0, \quad q\,\dot{\mathbf{r}} = \mathbf{0}, \tag{50}$$

and

$$m\,\ddot{\mathbf{r}} = -q\left(\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right) + q[\mathbf{r} \times (\nabla \times \mathbf{A})] - \mathfrak{B}\,\dot{\mathbf{r}}, \tag{51}$$

including losses due to friction, here called the dissipative Lorentz force.

4.2. Lagrangian density for an EM-field interacting with an elastic wave

In this second example, a stationary EM-field shall interact with an acoustic wave. The acoustic wave, which is an elastic wave, causes spatial strain variations in the medium which in turn generates spatial permittivity variations [28]. Hence, the EM-field is affected by this acoustic wave. The electromagnetic field is described by $\mathscr{L}_{\mathrm{EM}}$ and the acoustic field is described by $\mathscr{L}_{\mathrm{acoustic}}$

$$\mathscr{L} = \mathscr{L}_{\mathrm{EM}} + \mathscr{L}_{\mathrm{acoustic}} + \mathscr{L}_i. \tag{52}$$

A stationary EM-field with no x-component of the electric field is assumed, i.e. $\partial/\partial t = 0$ and $E_x = 0$. Therefore, its Lagrangian density is

$$\mathscr{L}_{\mathrm{EM}} = \frac{1}{2}\varepsilon(\nabla\varphi)^2 - \frac{1}{2\mu}(\nabla \times \mathbf{A})^2 - \varphi\varrho_{\mathrm{e}}. \tag{53}$$

The acoustic wave is a longitudinal wave propagating in x-direction. Potential and kinetic energies for an acoustic wave are given in [29] and, hence, the corresponding Lagrangian density is given as

$$\mathscr{L}_{\mathrm{acoustic}} = \frac{1}{2}\mathscr{B}\left(\frac{\partial\xi}{\partial x}\right)^2 - \frac{1}{2}\rho_{\mathrm{M}}\left(\frac{\partial\xi}{\partial t}\right)^2, \tag{54}$$

where

ξ is the longitudinal displacement of particles from their equilibrium position,
ρ_{M} is the specific mass of particles, and
$\mathscr{B}$ is the adiabatic bulk modulus.

The strain $\mathscr{S} = \partial\xi/\partial x$ causes a spatial variation $\Delta\varepsilon$ of the medium's permittivity. The permittivity is then $\varepsilon + \Delta\varepsilon$. For small variations in permittivity [28], the relation between $\Delta\varepsilon$ and $\mathscr{S} = \partial\xi/\partial x$ (tensor notation omitted) is

$$\Delta\varepsilon = -\varepsilon^2 P\frac{\partial\xi}{\partial x}, \tag{55}$$

with P as a dimensionless constant depending on the medium. This leads to the part of the Lagrangian density which accounts for the interaction, written as

$$\mathscr{L}_i = -\frac{1}{2}\varepsilon^2 P\frac{\partial\xi}{\partial x}(\nabla\varphi)^2. \tag{56}$$

The Lagrangian density $\mathscr{L} = \mathscr{L}_{\mathrm{EM}} + \mathscr{L}_{\mathrm{acoustic}} + \mathscr{L}_i$ obeys the Euler–Lagrange differential Eq. (3) with $\eta_1 = \varphi$, $\eta_2 = A_x$, $\eta_3 = A_y$, $\eta_4 = A_z$ and $\eta_5 = \xi$. Substituting $\eta_1 = \varphi$ and plugging $\mathscr{L}$ into Eq. (3) results in

$$\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)} = 0 \quad \text{since} \quad E_x = 0 \Rightarrow \frac{\partial\varphi}{\partial x} = 0\\
&\frac{\partial}{\partial y}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial y}\right)} = \frac{\partial}{\partial y}\left(\varepsilon\frac{\partial\varphi}{\partial y} - \varepsilon^2 P\frac{\partial\xi}{\partial x}\frac{\partial\varphi}{\partial y}\right) = -(\varepsilon + \Delta\varepsilon)\frac{\partial}{\partial y}E_y\\
&\frac{\partial}{\partial z}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial z}\right)} = \frac{\partial}{\partial z}\left(\varepsilon\frac{\partial\varphi}{\partial z} - \varepsilon^2 P\frac{\partial\xi}{\partial x}\frac{\partial\varphi}{\partial z}\right) = -(\varepsilon + \Delta\varepsilon)\frac{\partial}{\partial z}E_z\\
&\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} = 0\\
&\frac{\partial\mathscr{L}}{\partial\varphi} = -\varrho_{\mathrm{e}}\\
&\Rightarrow \nabla\cdot\mathbf{D} = \nabla\cdot(\varepsilon + \Delta\varepsilon)\mathbf{E} = \varrho_{\mathrm{e}}
\end{aligned} \tag{57}$$

It is seen that the acoustic wave affects Maxwell's first equation, i.e. Gauss' law. A modulation of strain in the medium results in a modulation of the electric field. Since $\mathscr{L}_i$ does not include the vector potential $\mathbf{A}$, the substitutions $\eta_2 = A_x$, $\eta_3 = A_y$ and $\eta_4 = A_z$ simply yield $\nabla \times \mathbf{H} = \mathbf{0}$, i.e. Ampère's law for a stationary EM-field. Substituting $\eta_5 = \xi$ and plugging $\mathscr{L}$ into Eq. (3) results in

$$\frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \xi}{\partial x})} = \frac{\partial}{\partial x}\left(\mathscr{B}\frac{\partial \xi}{\partial x} - \frac{1}{2}\varepsilon^2 P(\nabla\varphi)^2\right) = \frac{\partial}{\partial x}\mathscr{B}\frac{\partial \xi}{\partial x}$$
$$\frac{\partial}{\partial y}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \xi}{\partial y})} = 0$$
$$\frac{\partial}{\partial z}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \xi}{\partial z})} = 0$$
$$\frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial(\frac{\partial \xi}{\partial t})} = -\frac{\partial}{\partial t}\rho_{\mathrm{M}}\frac{\partial \xi}{\partial t}$$
$$\frac{\partial \mathscr{L}}{\partial \xi} = 0$$

$$\Rightarrow \frac{\partial^2 \xi}{\partial x^2} - \frac{\rho_{\mathrm{M}}}{\mathscr{B}}\frac{\partial^2 \xi}{\partial t^2} = 0, \tag{58}$$

which is the wave equation for the acoustic wave, whereas the speed of this acoustic wave is defined as $v = \sqrt{\mathscr{B}/\rho_{\mathrm{M}}}$. For $\eta_5 = \xi$, the interacting part of the Lagrangian density $\mathscr{L}_i$ does not contribute to Eq. (58), since $E_x = 0$ and therefore $\frac{\partial}{\partial x}\left(\frac{1}{2}\varepsilon^2 P(\nabla\varphi)^2\right) = 0$.

In general, it is very hard to find the interacting part of the Lagrangian density. However, with this simple example it has been demonstrated that it is possible to formulate fields of mixed type with a single Lagrangian density.

5. Conclusion

In this work, a Lagrangian density comprising electric and magnetic potentials has been developed. Furthermore, electric (ohmic) and magnetic losses are described by a dissipation density function.

It is now possible to derive all four Maxwell's equations in a straightforward way from a single, scalar Lagrangian density function and a single, scalar dissipation density function. Both functions satisfy the extended Euler–Lagrange differential equation.

The advantage of a Lagrangian formulation of time and space variant fields is not only its compactness but also the ability to formulate fields of mixed type. For instance, a single Lagrangian can describe acousto-optic systems where EM-fields interact with elastic waves.

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